UDC 62-50

THEOREM ON THE ALTERNATIVE FOR A NONSTATIONARY DIFFERENTIAL ENCOUNTER-EVASION GAME

PMM Vol. 42, No. 6, 1978, pp. 1123-1126

S. A. VAKHRAMEEV
(Moscow)

(Received April 5, 1978)

A position encounter-evasion differential game with non-stationary geometric constraints on the players' controls is analyzed. It is proved that the alternative is valid for this game, stating that either the position encounter game or the position evasion game is always solvable. The proof uses constructions analogous to the corresponding ones from [1].

1. Let the behavior of a controlled system Σ be described by the equation

$$x^{\cdot} = f(t, x, u, v), \quad u \in R^p, \quad v \in R^q \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are the controls of the first and second players, respectively. Let $\Omega(R^s)$ be the space of all nonempty compacta in R^s with the Hausdorff metric

$$h: \Omega(R^s) \times \Omega(R^s) \to R^1$$
$$h(A, B) = \min\{\varepsilon \geqslant 0 \mid A \subset B + S_\varepsilon,\ B \subset A + S_\varepsilon\}$$
$$S_\varepsilon = \{x \in R^s \mid |x| \leqslant \varepsilon\}, \quad |x| = \Big(\sum_{i=1}^{s} x_i^2\Big)^{1/2}$$

Let the measurable multivalued mappings (see [2, 3])

$$P: R^1 \to \Omega(R^p), \quad Q: R^1 \to \Omega(R^q)$$

be specified. We take it that at each instant t the players choose their own controls $u(t)$ and $v(t)$ from sets $P(t)$ and $Q(t)$, respectively.

We assume that the function $f(\cdot, x, u, v): R^1 \to R^n$ is measurable for all $x \in R^n$, $u \in R^p$, $v \in R^q$, while the function $f(t, \cdot, \cdot, \cdot): R^n \times R^p \times R^q \to R^n$ is continuous for any $t \in R^1$. By L_1^{loc} we denote the set of all locally Lebesgue - summable functions $g: R^1 \to R^1$. Let the inequality

$$| f(t, x, u, v) - f(t, y, u, v) | \leqslant \lambda(t) | x - y | \tag{1.2}$$

where the function $\lambda(\cdot) \in L_1^{\text{loc}}$ and is nonnegative, be fulfilled for all $x, y \in R^n$, $u \in P(t)$ and $v \in Q(t)$.

We assume that the inequality

$$| f(t, x, u, v) | \leqslant k(t)(1 + | x |) \tag{1.3}$$

is valid for all $x \in R^n, u \in P(t)$ and $v \in Q(t)$ where the function $k(\cdot) \in L_1^{\text{loc}}$ and is nonnegative.

We take it that the saddle point condition in the small game (see [1]) is fulfilled in the following form:

$$\max_{v\in Q(t)}\min_{u\in P(t)}(s,f(t,x,u,v)) = \min_{u\in P(t)}\max_{v\in Q(t)}(s,f(t,x,u,v)) \tag{1.4}$$

for any $x, s \in R^n$ and for almost all $t \in R^1$.

By $P(\cdot \mid t_1, t_2)$ we denote the set of all measurable branches of mapping $P: R^1 \to \Omega(R^p)$ on the halfopen interval $[t_1, t_2)$. By the theorem on measurable selectors (see [2, 3]) this set is not empty. A mapping which associates a nonempty set from $P(\cdot \mid t, \infty)$ with an arbitrary position (t, x) is called the first player's strategy $U \div U(\cdot \mid t, x)$. The symbol $Q(\cdot \mid t_1, t_2)$ and the second player's strategy $V \div V(\cdot \mid t, x)$ are defined analogously.

Suppose that the first player has chosen a strategy $U \div U(\cdot \mid t, x)$. We consider a partitioning Δ of the semiaxis $[t_0, \infty)$ into a system of half-open intervals of the form $\tau_i \leqslant t < \tau_{i+1}$, $i = 0, 1, 2, \ldots$, $t_0 = \tau_0$, $\tau_i \to \infty$ as $i \to \infty$. We denote $|\Delta| = \sup_i (\tau_{i+1} - \tau_i)$. Let $v(\cdot) \in Q(\cdot \mid t_0, \infty)$ be an arbitary realization of the second player's operations. We consider the ordinary differential equation

$$\begin{aligned} &x^\cdot = f(t, x, u_i(t), v(t)), \quad \tau_i \leqslant t < \tau_{i+1} \\ &u_i(\cdot) \in U(\cdot \mid \tau_i, x(\tau_i)), \quad i = 0, 1, 2, \ldots \\ &x(t_0) = x_0 \end{aligned}$$

This equation has a solution $x(t) = x(t; t_0, x_0, U, v(\cdot))$, continuable onto $[t_0, \infty)$, which is called the Euler polygonal line generated by the first player's strategy $U \div U(\cdot \mid t, x)$.). Every function $x(\cdot)$ for which we can find, on any finite interval $t_0 \leqslant t \leqslant t_1$, a sequence $\{x_k(\cdot)\}$ of Euler polygonal lines $x_k(t) = x_k(t;\ t_0, x_0^k, U, v_k(\cdot))$, such that it converges uniformly to $x(\cdot)$ on the interval $t_0 \leqslant t \leqslant t_1$ as $|\Delta^{(k)}| \to 0$, $x_0^k \to x_0$, $k \to \infty$, is called a motion $x(t) = x(t; t_0, x_0, U)$ generated by the first player's strategy $U \div U(\cdot \mid t, x)$. The motion generated by the second player's strategy $V \div V(\cdot \mid t, x)$ is defined analogously.

Let the nonempty closed sets M_c and N_c be prescribed in position space R^{n+1}. The encounter-evasion game is put together from the following two problems.

P r o b l e m 1. Find the strategy $U^c \div U^c(\cdot \mid t, x)$ which ensures the contact

$$\begin{aligned} &(t, x(t)) \in N_c, \quad t_0 \leqslant t < \tau, \quad (\tau, x(\tau)) \in M_c \\ &(t, x(t)) \notin M_c, \quad t_0 \leqslant t < \tau \end{aligned}$$

for all motions $x(t) = x(t; t_0, x_0, U^c)$.

P r o b l e m 2. Find open neighborhoods $H(N_c)$ and $G(M_c)$ of sets N_c and M_c and the strategy $V^c \div V^c(\cdot \mid t, x)$, such that the contact $(t, x(t)) \in H(N_c)$, $t_0 \leqslant t < \tau$, $(\tau, x(\tau)) \in G(M_c)$ is excluded for all motions $x(t) = x(t; t_0, x_0, V^c)$.

When the set M_c lies wholly in the halfspace $\{t \in R^1 \mid t \leqslant T\}$ we speak of Problem 1 as the problem of encounter with set M_c inside set N_c by the instant T and we speak of Problem 2 as the problem of evading M_c inside N_c up to the instant T.

2. Let conditions (1.2) and (1.3) be fulfilled. We consider the differential inclusion

$$x^\cdot \in \operatorname{conv}\{f(t, x, u, v);\ u \in P(t),\ v \in Q(t)\} \tag{2.1}$$

It obviously has a solution $x(\cdot)$, continuable onto $[t_0, \infty)$, satisfying the initial condition $x(t_0) = x_0$. It can be shown that the inequality

$$|f(t, x(t), u, v)| \leqslant m^0(t) \tag{2.2}$$

for all $u \in P(t)$ and $v \in Q(t)$ is valid for any solution $x(\cdot)$ of inclusion (2.1). Bound (2.2) is uniform for all positions (t_0, x_0) from some bounded domain G of space R^{n+1}. Here the function $m^0(\cdot) \in L_1^{\mathrm{loc}}$ and depends only on domain G.

The nonstationarity of the constraints on the player's controls leads to the following modification of the definition of stability (see [1]). We say that a set $W \subset R^{n+1}$ is u-stable if for any position $(t_*, x_*) \in W$, any instant $t^* > t_*$ and any control $v^*(\cdot) \in Q(\cdot \mid t_*, t^*)$ of the second player there exists a solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of the inclusion

$$x^{\cdot} \in \operatorname{conv}\{f(t, x, u, v^*(t));\ u \in P(t)\}$$

with initial condition $x(t_*) = x_*$, such that $(t^*, x(t^*)) \in W$ or $(\tau^*, x(\tau^*)) \in M$ for some $\tau^* \in [t_*, t^*]$. We say that a set $W \subset R^{n+1}$ is v-stable if for any position $(t_*, x_*) \in W$, and instant $t^* > t_*$ and any control $u^*(\cdot) \in P(\cdot \mid t_*, t^*)$ of the first player there exists a solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of the inclusion

$$x^{\cdot} \in \operatorname{conv}\{f(t, x, u^*(t), v;\ v \in Q(t)\}$$

with initial condition $x(t_*) = x_*$, such that $(t^*, x(t^*)) \in W$ or $(\tau^*, x(\tau^*)) \notin H(N_c)$ for some $\tau^* \in [t_*, t^*]$. The property of u-stability (v-stability) of set W is defined with respect to a prescribed closed set M_c (with respect to a prescribed open neighborhood $H(N_c)$ of a prescribed set N_c). Such a definition of stability properties, differing from the analogous definition in [1], does not alter the following important property.

Lemma 1. If set W is u-stable (v-stable), then its closure $\overline{W}$ is u-stable (v-stable).

We now define the first player's extremal strategy $U^e \div U^e(\cdot \mid t, x)$. Let (t_*, x_*) be an arbitrary position and let set $W \subset R^{n+1}$ be closed. We consider the hyperplane $\Gamma_{t_*} = \{(t, x) \in R^{n+1} \mid t = t_*\}$. If $\Gamma_{t_*} \cap W = \phi$, then we set $U^e(\cdot \mid t_*, x_*) = P(\cdot \mid t_*, \infty)$; if $\Gamma_{t_*} \cap W \neq \phi$, then by ω_* we denote the vector of the section $W(t_*)$ of set W by hyperplane Γ_{t_*}, which lies closest to (t_*, x_*). Then

$$U^e(\cdot \mid t_*, x_*) = \{u^*(\cdot) \in P(\cdot \mid t_*, \infty \mid \max_{v \in Q(t)} (x_* - \omega_*, f(t, x_* u^*(t), v)) = \min_{u \in P(t)} \max_{v \in Q(t)} (x_* - \omega_*, f(t, x_*, u, v))\}$$

The second player's extremal strategy is defined analogously. Namely, if $\Gamma_{t_*} \cap W = \phi$, then we set $V^e(\cdot \mid t_*, x_*) = Q(\cdot \mid t_*, \infty)$; if $\Gamma_{t_*} \cap W \neq \phi$, then

$$V^e(\cdot \mid t_*, x_*) = \{v^*(\cdot) \in Q(\cdot \mid t_*, \infty \mid \min_{u \in P(t)} (\omega_* - \omega_*, f(t, x_*, u, v^*(t))) = \max_{v \in Q(t)} \min_{u \in P(t)} (\omega_* - x_*, f(t, x_*, u, v))\}$$

The symbols ω_* and Γ_{t_*} here have the same meaning as above. Using Filippov's theorem (see [3]), it is easy to prove that these definitions are well posed.

3. Let the function $x(t)$, $t \geqslant t_*$, satisfy the equation

$$x^{\cdot} = f(t, x, u^*(t), v(t)), \quad x(t_*) = x_*$$

and let the function $y(t)$, $t \geqslant t_*$, satisfy the differential inclusion

$$y^{\cdot} \in \operatorname{conv} \{f(t, y, u, v^*(t));\ u \in P(t)\}, \qquad y(t_*) = y_*$$

Here the function $v(\cdot) \in Q(\cdot \mid t_*, \infty)$ is arbitrary, while $u^*(\cdot) \in P(\cdot \mid t_*, \infty)$ and $v^*(\cdot) \in Q(\cdot \mid t_*, \infty)$ have been chosen from the conditions

$$s_* = x_* - y_*$$
$$\max_{v \in Q(t)} (s_*, f(t, x_*, u^*(t), v)) = \min_{u \in P(t)} \max_{v \in Q(t)} (s_*, f(t, x_*, u, v))$$
$$\min_{u \in P(t)} (s_*, f(t, x_*, u, v^*(t))) = \max_{v \in Q(t)} \min_{u \in P(t)} (s_*, f(t, x_*, u, v))$$

We denote $\rho(t) = |x(t) - y(t)|$.

Lemma 2. The following estimate:

$$\rho^2(t) \leqslant \rho^2(t_*) \left(1 + 2\int_{t_*}^{t} \lambda(\xi)\, d\xi\right) + \int_{t_*}^{t} \varphi(t_*, \xi)\, m(\xi)\, d\xi \tag{3.1}$$

is valid and is uniform for all (t_*, x_*) and (t_*, y_*) from some bounded domain $G \subset R^{n+1}$. Here

$$m(t) = 4g\lambda(t) + 8m^\circ(t), \; g = \operatorname{diam} G, \; \varphi(t_*, t) = \int_{t_*}^{t} m^\circ(\xi)\, d\xi$$

where the function $m^\circ(\cdot)$ is from (2.2) and function $\lambda(\cdot)$ is from (1.2).

The proof of this statement differs only in certain details from that of the analogous statement in [1].

4. The following barrier properties of extremal strategies enable us to prove the theorem on the alternative.

Lemma 3. Let $W \subset R^{n+1}$ be a closed u-stable set, $U^e \div U^e(\cdot \mid t, x)$ be an extremal strategy and $(t_0, x_0) \in W$. Then the inclusion $(t, x(t)) \in W$ is fulfilled for any motion $x(t) = x(t; t_0, x_0, U^e)$ up to the contact $(\tau, x(\tau)) \in M_c$.

This statement can be proved in the same way as the analogous statement in [1], except only that instead of the auxiliary bound (15.1) in [1] we need to use the inequality

$$\varepsilon_k^2(t) \leqslant \left(\varepsilon_k^2(t_*) + \varphi_k \int_{t_*}^{t} m(s)\, ds\right) \exp\left(2\int_{t_*}^{t} \lambda(s)\, ds\right)$$
$$t_* \leqslant t \leqslant t_* + \tau_*, \qquad \varphi_k = \sup_i \sup_{\tau_i^{(k)} \leqslant t < \tau_{i+1}^{(k)}} \varphi(\tau_i^{(k)}, t), \; k = 1, 2, \ldots$$

where $\tau_i^{(k)}$ are the points of partitioning $\Delta^{(k)}$ corresponding to the polygonal line $x_k(t) = x_k(t;\ t_0, x_0^k, U^e, v_k(\cdot))$. This inequality is a direct consequence of the u-stability of set W and of Lemma 2. Completely analogously we obtain

Lemma 4. If a closed set $W \in R^{n+1}$ is v-stable, $V^e \div V^e(\cdot \mid t, x)$ is an extremal strategy and $(t_0, x_0) \in W$, then the inclusion $(t, x(t)) \in W$ is fulfilled for any motion $x(t) = x(t;\ t_0, x_0, V^e)$ up to the instant τ, when $(\tau, x(\tau)) \notin H(N_0)$.

The use of Lemmas 3 and 4 and a literal repetition of the arguments in Sections 16 and 17 of [1] lead to the following theorem.

T h e o r e m 1. Let an initial position (t_0, x_0) be given and an instant $T \geqslant t_0$ be chosen. Then either a strategy $U^c \div U^c(\cdot \mid t, x)$ exists, solving Problem 1 on encounter with M_c inside N_c by the instant T, or open neighborhoods $H(N_c)$ and $G(M_c)$ of sets N_c and M_c and a strategy $V^c \div V^c(\cdot \mid t, x)$ exist, solving Problem 2 on evading M_c inside N_c up to instant T.

N o t e. The constructions presented are not changed if we use single-valued strategies instead of multivalued ones, i.e., if we define the first (second) player's strategy as a mapping which associates a certain function from $P(\cdot \mid t, \infty)$ $(Q(\cdot \mid t, \infty))$ with a arbitrary position (t, x) .

In conclusion the author thanks N. N. Krasovskii, A. I. Subbotin and M. S. Nikol'skii for attention to the work.

REFERENCES

1. K r a s o v s k i i, N. N. and S u b b o t i n, A. I., Position Differential Games. Moscow, "Nauka", 1974.

2. I o f f e, A. D. and T i k h o m i r o v, V. M., Theory of Extremal Problems, Moscow, "Nauka", 1974.

3. W a r g a, J., Optimal Control of Differential and Functional Equations. Moscow, "Nauka", 1977.

Translated by N. H. C.